Introduction

If you buy 4 apples for a total cost of 80 pence, how much does each apple cost?

$$Cost\ of\ one\ apple = \frac{Total\ cost}{Number} = \frac{80\ pence}{4} = 20\ pence$$

In maths we often use symbols to represent quantities. Here the cost of one apple could be denoted by $\boldsymbol{a}$.

We could rewrite this as:

$$a = \frac{T}{N}$$

The real power of algebra is that we can apply this formula to anything.
$\boldsymbol{a}$ could be the cost of bananas or cherries.

Algebra allows us to move from the specific to the general, the technique for one situation helping us cope with many.

Here are some of the formulae that have allowed us to become sophisticated users of technology, from building bridges and mobile phones to getting to the moon:

$$\boldsymbol{F = ma}$$

$$\boldsymbol{V = IR}$$

$$\boldsymbol{P = IV}$$

$$\boldsymbol{v = f\lambda}$$

$$curl\boldsymbol{B} = \mu_0\left(\boldsymbol{j}_f + curl\boldsymbol{M} + \frac{\partial \boldsymbol{P}}{\partial t}\right) + \varepsilon_0\mu_0\frac{\partial \boldsymbol{E}}{\partial t}$$

Challenge:

Find out what these formulae are for and who discovered them.

Basic rules of algebra

1. The letter x is often used to denote an unknown quantity.

However, a, b, c, … are just as good and sometimes we use letters (Variables) from other alphabets, such as: α, β, π, θ.

2. One important example of algebraic shorthand is

$3a = 3 \times a \qquad 7y = 7 \times y \qquad ab = a \times b \qquad and\ so\ on$

3. BIDMAS

This gives the order in which to carry out the mathematical operations.

Brackets

Indices (e.g. squares, cubes, square roots etc)

Division

Multiplication

Addition

Subtraction

Challenge:

In English we read and write from left to right. Find out which languages read and write in other directions.

Examples

A. $8 - 3 \times 2$

It may seem natural to do this as $8 - 3 = 5$ and then $5 \times 2 = 10$. But this is wrong!
BIDMAS means do Multiplication before Subtraction.
$8 - 3 \times 2 = 8 - 6 = 2$ is correct.

B. Calculate 25% of £8

$$25\%\ of\ 8 = \frac{25}{100} \times 8 = \frac{1}{4} \times 8 = 2$$

This is Division first then Multiplication. However, you could also do…

$$25\%\ of\ 8 = \frac{25}{100} \times 8 = \frac{25 \times 8}{100} = \frac{200}{100} = 2$$

This is Multiplication first then Division. It doesn't matter which order you do multiply and divide. BIDMAS = BIMDAS!

Addition and Subtraction can be also be swapped. BIDMAS = BIDMSA!

4. 'Like terms' can be grouped together.

$$a + c + b + a + b + a = 3a + 2b + c$$

Normally we write the letters in alphabetical order

This is just like rearranging objects.

is the same as

5. Brackets can be used as a way of expressing multiplication

Suppose $v = at + 5$ and $a = 4, t = 8$

$$\rightarrow \quad v = (4)(8) + 5 = 4 \times 8 + 5 = 37$$

Brackets are especially useful when dealing with negative numbers.

6. Indices (or 'powers')

$$x^1 = x \qquad x^2 = x \times x \qquad x^3 = x \times x \times x \qquad x \times x^2 = x^3$$

$$a^3 \times a^2 = a \times a \times a \;\times\; a \times a \;=\; a^5$$

$$4bc^2a^3 \times 5ac^3 = 4 \times 5 \times a \times a \times a \times a \times b \times c \times c \times c \times c \times c$$
$$= 20a^4bc^5$$

$$\frac{b^4a^3c}{c^2a^2b} = \frac{a \times a \times a \times b \times b \times b \times b \times c}{a \times a \times b \times c \times c} = \frac{ab^3}{c}$$

7a. Multiplying single brackets

$$3(a+b) = 3a + 3b$$

$$x(a-b) = ax - bx$$

$$\begin{aligned} -3(2x+4) &= -3 \times 2x + (-3) \times 4 \\ &= -6x + (-12) \\ &= -6x - 12 \end{aligned}$$

$$\begin{aligned} -x(4y-5) &= -x \times 4y - (-x) \times 5 \\ &= -4xy - (-5x) \\ &= -4xy + 5x \\ &= 5x - 4xy \end{aligned}$$

$$\begin{aligned} 3(4a-9) - 5(2-a) &= 12a - 27 - 10 + 5a \\ &= 12a + 5a - 27 - 10 \\ &= 17a - 37 \end{aligned}$$

By multiplying out the brackets and combining like terms we have an expression that is a lot simpler than the original.

Let's simplify $-x(5x-2) + x(x^2+1)$:

$$\begin{aligned} -x(5x-2) + x(x^2+1) &= -5x^2 + 2x + x^3 + x \\ &= x^3 - 5x^2 + 3x \end{aligned}$$

(This is a *polynomial*.)

It is normal to write these in descending order of index (power).

Exercise A

Multiply these brackets and simplify where possible.

1. $7(a+4)$
2. $x(y-3)$
3. $-3(b+2)+5$
4. $(a+5)h$
5. $-2x(x-5)$
6. $-(g-3)$
7. $3(4a-9)-a(2-a)$
8. $-x(4-x)-x^2(x-a)$

7b. Multiplying double brackets

Each term in the first bracket multiplies each term in the second.

Method A

$$(a+b)(x+y) = ax + ay + bx + by$$

Check:

$$(2+4)(3+8) = 2\times 3 + 2\times 8 + 4\times 3 + 4\times 8$$
$$= 6 + 16 + 12 + 32 = 66$$
$$(2+4)(3+8) = 6\times 11 = 66 \checkmark$$

Method B

$$(a+b)(x+y) = a(x+y) + b(x+y)$$
$$= ax + ay + bx + by$$

Always be careful with negative signs.

$$(x - 2)(3 - a) = 3x - ax - 6 + 2a$$

$$\begin{aligned}(x + 2)(x - 6) &= x \times x - 6x + 2x - 12 \\ &= x^2 - 6x + 2x - 12 \\ &= x^2 - 4x - 12\end{aligned}$$

$$\begin{aligned}(a - 3)^2 &= (a - 3)(a - 3) \\ &= a^2 - 3a - 3a + 9 \\ &= a^2 - 6a + 9\end{aligned}$$

$$\begin{aligned}(c + 3)(4 - c) &= 4c - c^2 + 12 - 3c \\ &= -c^2 + c + 12\end{aligned}$$

Exercise B

Multiply these brackets and simplify where possible.

1. $(a + 5)(a + 4)$
2. $(y + 6)(y - 3)$
3. $(b - 5)(b + 2)$
4. $(-a - 5)(a - 8)$
5. $(x - 2)^2 + 3(x - 1)$
6. $(4a - 9)(2 - a)$
7. $(x - 7)(x^2 - 4x + 2)$
8. $(2k + 3)(3k^2 + 4k - 6)$

9. Factorising

Numbers that multiply to give another number are called factors.

The factors of 60 are 60, 30, 20, 15, 12, 10, 6, 5, 4, 3, 2, 1

Factorising is the process of breaking numbers into factors. All natural numbers (1, 2, 3 ...) can be expressed as a product of prime factors.

e.g. $60 = 5 \times 3 \times 2 \times 2$

We can factorise expressions involving letters as well as numbers:

$$a^3 = a^2 \times a = a \times a \times a$$

$$4x^2 = 4x \times x = 2x \times 2x \ \ etc$$

$$x^2y = x^2 \times y = x \times xy$$

Common factors

It is an important skill to spot common factors.

Examples

$18 \ = \ 18 \times 1 \ = \ 9 \times 2 \ = \ 6 \times 3$

$24 \ = \ 24 \times 1 \ = \ 12 \times 2 \ = \ 8 \times 3 \ = \ 6 \times 4$

So the common factors of 18 and 24 are 2 and 3 and 6.

This is useful when cancelling down fractions:

$$\frac{18}{24} = \frac{3 \times 6}{4 \times 6} = \frac{3}{4}$$

By cancelling out the largest common factor we get the simplest form.

When factorising, look for the Highest Common Factor.

$$3a + 3b = 3(a + b)$$

3 is the highest common factor

This is just the reverse process to multiplying out the brackets.

After identifying the highest common factor put it out front:

$$3a + 3b = 3(\quad + \quad)$$

The missing bits in the bracket have to make up to the original.

$$12a + 6b = 6(2a + b)$$

6 is the highest common factor

Always check by multiplying back out:

$$6(2a + b) = 12a + 6b \checkmark$$

$$15c - 5cd = 5c(3 - d)$$

$5c$ is the highest common factor

$$14g^2h - 21gh = 7gh(2g - 3)$$

$$3x^2y + x^2 = x^2(3y + 1)$$

Exercise C

Factorise these then check your answer by multiplying out.

1. $12a - 18b$
2. $20c + 10c^3d$
3. $-6ef^2 - 3$
4. $8g^2h^2 + 6gh^3$
5. $3i - 27ij^2$
6. $50k^5l^3 - 15k^2l^3$
7. $20m + 2mn - 12m^2$
8. $\frac{1}{2}p^2q + \frac{3}{2}p^3q^3$