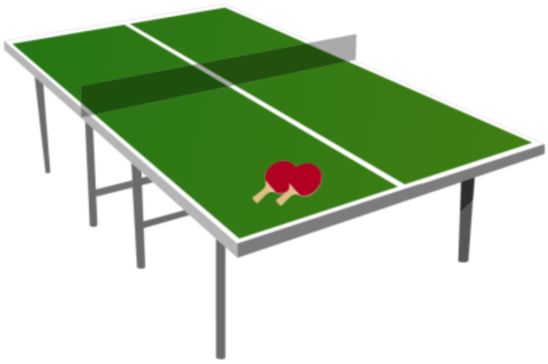

I play table tennis against a friend. The probability I win a point is $\boldsymbol{a}$ and the probability she wins a point is $\boldsymbol{b}$. These probabilities stay constant throughout the game.

(Note: $\boldsymbol{a} \neq \boldsymbol{b}$)

A game is won only when a player wins two consecutive points.

So, for example, I might win a game where the point rallies go: WLWLWLWLWW (that is, I win the game because I won the final two points).

In the following, simplify your answers wherever possible.

1. Write a simple relationship between $\boldsymbol{a}$ and $\boldsymbol{b}$.

2. What is the probability I win the first two points (and so win the game)?

3. What is the probability the game lasts for three points and I win?

4. What is the probability the game lasts for four points and I win?

   (You may like to repeat question 4 for when the game lasts for five points, or six, or seven and so on.)

5. Describe mathematically any patterns you find in these probabilities.

6. What is the probability, in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$, that the game lasts for N points and I win?

7. If $\boldsymbol{a} = 0.6$, show that the probability I win a game in five points or less is approximately 0.625

8. Write a ***proof*** or ***justification*** of your answer to question 6.

Teacher notes

This task could be given to high-achieving GCSE students or to A Level students. It was originally designed for high-achieving MYP students.

Solutions

In the following, W *stands for a point won by me,* L *for a point won by my opponent*

1. $a + b = 1$
2. $p(I\ win\ first\ two\ points) = a \times a = a^2$ (The events are independent, so we simply multiply individual probabilities.)
3. For this to happen, the points sequence must go: LWW

   $\Rightarrow p(I\ win\ a\ three\ point\ game) = a^2b$

4. For this to happen, the points sequence must go: WLWW

   $\Rightarrow p(I\ win\ a\ four\ point\ game) = a^3b$

   Extending this (to have enough data to formulate *p(I win an N-point game)*):

   To win a five point game, the points sequence must go: LWLWW

   $\Rightarrow p(I\ win\ a\ five\ point\ game) = a^3b^2$

   To win a six point game, the points sequence must go: WLWLWW

   $\Rightarrow p(I\ win\ a\ six\ point\ game) = a^4b^2$

   To win a seven point game, the points sequence must go: LWLWLWW

   $\Rightarrow p(I\ win\ a\ seven\ point\ game) = a^4b^3$

   To win an eight point game, the points sequence must go: WLWLWLWW

   $\Rightarrow p(I\ win\ an\ eight\ point\ game) = a^5b^3$

5. There are a number of patterns on display, such as:
   - the total of the powers of a and b is equal to the number of points played
   - for me to win an even-point game, I must win the first point
   - for me to win an odd-point game, I must lose the first point
   - when I win an even-point game I outscore my opponent by two points
   - when I win an odd-point game I outscore my opponent by one point.

There are actually two sets of patterns - one for when N is even, and one for when N is odd (see answer to question 6 below).

6. To win an N-point game, it depends if N is even or odd:

N even $\Rightarrow p(I\ win\ an\ N-point\ game) = a^{\frac{N}{2}+1}b^{\frac{N}{2}-1}$

N odd $\Rightarrow p(I\ win\ an\ N-point\ game) = a^{\frac{N+1}{2}}b^{\frac{N-1}{2}}$ [Note: $N \geq 2$]

7. The probability I win a game in five points or less is:

$= p(I\ win\ in\ 2\ points) + p(I\ win\ in\ 3) + p(I\ win\ in\ 4) + p(I\ win\ in\ 5)$

$= a^2 + a^2b + a^3b + a^3b^2$

If $a = 0.6$ and $b = 0.4$ this gives a probability of

$(0.6)^2 + (0.6)^2(0.4) + (0.6)^3(0.4) + (0.6)^3(0.6)^2 = 0.62496 = 0.625$ (3sf)

8. Proof of answer(s) to question 6:

In the case when N is even, I need to win two more points than my opponent.

This means that if I win x points, my opponent wins $N - x$, and so $x - (N - x) = 2$

$\Rightarrow\ 2x - N = 2 \Rightarrow\ x = \frac{N+2}{2} = \frac{N}{2} + 1$ and $N - x = N - \left(\frac{N}{2} + 1\right) = \frac{N}{2} - 1$

These are independent events, so $p(I\ win) = a^x b^{N\text{-}x} = a^{\frac{N+1}{2}}b^{\frac{N-1}{2}}$

In the case when N is odd, I need to win 1 more point than my opponent.

Again, if I win x points, my opponent wins $N - x$, and so $x - (N - x) = 1$

$\Rightarrow\ 2x - N = 1\ \Rightarrow\ x = \frac{N+1}{2}$ and $N - x = N - \left(\frac{N+1}{2}\right) = \frac{N-1}{2}$

These are still independent events, so $p(I\ win) = a^x b^{N\text{-}x} = a^{\frac{N+1}{2}}b^{\frac{N-1}{2}}$

Another possible justification:

If my opponent and I are equally matched, then $a = b = 0.5$

N even $\Rightarrow p(I\ win\ an\ N-point\ game) = a^{\frac{N}{2}+1}b^{\frac{N}{2}-1} = a^{\frac{N}{2}+1}a^{\frac{N}{2}-1} = a^N = 0.5^N$

N odd $\Rightarrow p(I\ win\ a\ N-point\ game) = a^{\frac{N+1}{2}}b^{\frac{N-1}{2}} = a^{\frac{N+1}{2}}a^{\frac{N-1}{2}} = a^N = 0.5^N$

If we are evenly matched, the above are the probabilities that my opponent wins.

So, probability I win in 2 or 3 or 4 or … games is:

$(0.5)^2 + (0.5)^3 + (0.5)^4 + (0.5)^5 + \ \dots = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \dots$ which tends to 0.5

And the probability my opponent wins in 1 or 2 or 3 or 4 or …. games is the same.