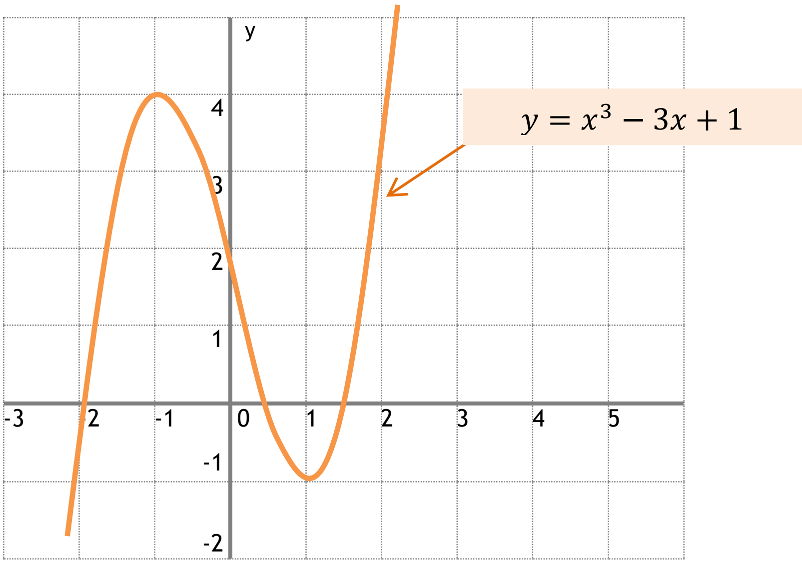
The roots of the equation $x^3 - 3x + 1 = 0$ are the values of x where the curve

$f(x) = x^3 - 3x + 1$ crosses the $x - axis$



The above curve shows us there are 3 roots to the equation $x^3 - 3x + 1 = 0$ and that $f(x)$ changes sign either side of a root. We can check for roots algebraically using this idea.

In the interval $1 < x < 2$

$f(1) = 1^3 - 3(1) + 1 = -1 < 0$

$f(2) = 2^3 - 3(2) + 1 = 3 > 0$

$\Rightarrow$ there is a root between $x = 1$ and $x = 2$, where $f(x) = 0$

Searching for the roots by iteration

Arrangement 1	Arrangement 2	Arrangement 3
$x^3 - 3x + 1 = 0$	$x^3 - 3x + 1 = 0$	$x^3 - 3x + 1 = 0$
$x^3 + 1 = 3x$	$x^3 = 3x - 1$	$x^3 = 3x - 1$
$x = \frac{1}{3}(x^3 + 1)$	$x = (3x - 1)^{\frac{1}{3}}$	$x^2 = 3 - \frac{1}{x}$
Let $x_{n+1} = \frac{1}{3}({x_n}^3 + 1)$	Let $x_{n+1} = (3x_n - 1)^{\frac{1}{3}}$	Let $x_{n+1} = \sqrt{3 - \frac{1}{x_n}}$

Searching for the root in the interval $1 < x < 2$, starting with $x_1 = 1.5$

Arrangement 1

Let $x_{n+1} = \frac{1}{3}({x_n}^3 + 1)$

$x_1 = 1.5$

$x_2 = 1.45833..$

$x_3 = 1.3671633..$

$x_4 = 1.18513797..$

$x_5 = 0.888195 \ldots$

$x_6 = 0.566896 \ldots$

$x_7 = 0.394061 \ldots$

$x_8 = 0.353730 \ldots$

$x_9 = 0.348086 \ldots$

Arrangement 2

Let $x_{n+1} = (3x_n - 1)^{\frac{1}{3}}$

$x_1 = 1.5$

$x_2 = 1.518294 \ldots$

$x_3 = 1.526189 \ldots.$

$x_4 = 1.529571 \ldots..$

$x_5 = 1.531015 \ldots..$

$x_6 = 1.531631 \ldots..$

$x_7 = 1.531894 \ldots..$

$x_8 = 1.532005 \ldots..$

Arrangement 3

Let $x_{n+1} = \sqrt{3 - \frac{1}{x_n}}$

$x_1 = 1.5$

$x_2 = 1.5275225 \ldots$

$x_3 = 1.5314523 \ldots$

$x_4 = 1.5320003 \ldots$

$x_5 = 1.532076 \ldots$

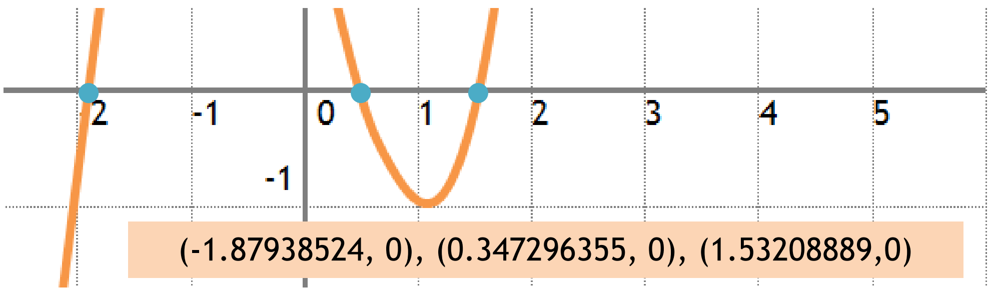
$x_6 = 1.5320871 \ldots$

$x_7 = 1.5320886 \ldots$

Arrangement 1 is closing in on the root between $x = 0$ and $x = 1$!

Arrangement 2 is closing in on the root between $x = 1$ and $x = 2$, as hoped for!

Arrangement 3 is also closing in on the root between $x = 1$ and $x = 2$ and faster than arrangement 2



Ok, so we have 'found' 2 of the roots, so let's search for the root in the interval $-2 < x < -1$

Arrangement 1	Arrangement 2	Arrangement 3
Let $x_{n+1} = \frac{1}{3}({x_n}^3 + 1)$	Let $x_{n+1} = (3x - 1)^{\frac{1}{3}}$	Let $x_{n+1} = \sqrt{3 - \frac{1}{x_n}}$
$x_1 = -1.5$	$x_1 = -1.5$	$x_1 = -1.5$
$x_2 = -0.7916666..$	$x_2 = -1.765174 \ldots$	$x_2 = 1.9148542 \ldots.$
$x_3 = 0.16794463..$	$x_3 = -1.8464771 \ldots.$	$x_3 = 1.57409244 \ldots$
$x_4 = 0.3349123 \ldots$	$x_4 = -1.87002176 \ldots..$	$x_4 = 1.5377624 \ldots$
$x_5 = 0.345855 \ldots$	$x_5 = -1.8767305 \ldots..$	This one is heading towards the largest root
This is heading towards the middle root!	$x_6 = -1.87863333 \ldots..$	
	$x_7 = -1.87917234 \ldots..$	
	$x_8 = -1.8793249 \ldots..$	
	This is heading in the right direction	

The above three iterations show the unpredictable nature of this technique!

Looking for turning points

$f(x) = 2e^x - 3\ln(x) - 4,\ x > 0$

Find $f'(x)$

The turning points are when $f'(x) = 0$

Rearrange $f'(x) = 0$ to give the two iterations

- $x_{n+1} = \frac{3}{2e^{x_n}}$
- $x_{n+1} = ln\left(\frac{3}{2x_n}\right)$

Show that the turning point is in the interval $0.4 < x < 1$

Search for the turning point using each of the two iterations, starting with $x_1 = 0.5$

Teacher notes

Looking for turning points

$f(x) = 2e^x - 3\ln(x) - 4,\ x > 0$

$f'(x) = 2e^x - \frac{3}{x}$

$f'(0.4) < 0$ and $f'(1) > 0 \Rightarrow$ there is a turning point in the interval $0.4 < x < 1$

Arrangement 1

$2e^x - \frac{3}{x} = 0$

$2xe^x - 3 = 0$

$2xe^x = 3$

$x = \frac{3}{2e^x}$

$x_{n+1} = \frac{3}{2e^{x_n}}$

$x_1 = 0.5$

$x_2 = 0.9097959\ ...$

$x_3 = 0.6039096\ ...$

$x_4 = 0.8200053\ ...$

$x_5 = 0.6606439..$

$x_6 = 0.7747779..$

$x_7 = 0.6912091\ .....$

$x_8 = 0.7514549..$

$x_9 = 0.7075196..$

This iteration is oscillating slowly towards the turning point (0.7259, 1.0942)

Arrangement 2

$2e^x - \frac{3}{x} = 0$

$2xe^x - 3 = 0$

$2xe^x = 3$

$e^x = \frac{3}{2x}$

$x_{n+1} = ln\left(\frac{3}{2x_n}\right)$

$x_1 = 0.5$

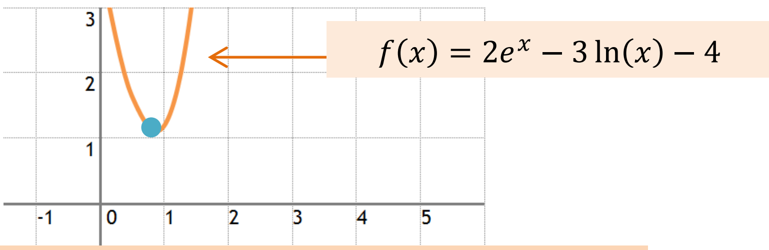
$x_2 = 1.0986122\ ...$

$x_3 = 0.31141728\ ...$

$x_4 = 1.5720866\ ...$

$x_5 = -0.04693\ ....$

This iteration leads to the logarithm of a negative value, which does not exist for the set of real numbers.



Turning point (0.725908074, 1.09420944)

A spreadsheet is a good way to apply iterations.

	A	B
1	x	3/(2e^x)
2	0.500000000000000	0.909795989568950
3	0.909795989568950	0.603909527326859
4	0.603909527326859	0.820005346011644
5	0.820005346011644	0.660643949939309
6	0.660643949939309	0.774777922867482
7	0.774777922867482	0.691209156202476
8	0.691209156202476	0.751454927655335
9	0.751454927655335	0.707519689940667
10	0.707519689940667	0.739297711564400
11	0.739297711564400	0.716173657188550
12	0.716173657188550	0.732927457124984
13	0.732927457124984	0.720750427726912
14	0.720750427726912	0.729580680887924
15	0.729580680887924	0.723166659168934
16	0.723166659168934	0.727819973102637
17	0.727819973102637	0.724441065935519
18	0.724441065935519	0.726893025183865
19	0.726893025183865	0.725112896401682
20	0.725112896401682	0.726404840311002
21	0.726404840311002	0.725466971968959
22	0.725466971968959	0.726147683634231
23	0.726147683634231	0.725653554634039

	A	B
1	x	3/(2e^x)
2	0.5	=3/(2*EXP(A2))
3	=B2	=3/(2*EXP(A3))
4	=B3	=3/(2*EXP(A4))
5	=B4	=3/(2*EXP(A5))
6	=B5	=3/(2*EXP(A6))
7	=B6	=3/(2*EXP(A7))
8	=B7	=3/(2*EXP(A8))