Find the roots of these quadratic equations – they all factorise

1. $x^2 + 6x + 8 = 0$
2. $x^2 - 5x + 6 = 0$
3. $2x^2 + 7x - 15 = 0$
4. $3x^2 - 11x + 10 = 0$
5. $x^2 - 16 = 0$

Let the roots of the quadratic equation $ax^2 + bx + c = 0$ be α and β

For each of the above equations

6. What is the connection between the coefficients a, b and c, the sum of the roots, $\alpha + \beta$ and the product of the roots, $\alpha\beta$?
7. What type of number is the value of the discriminant $b^2 - 4ac$?
8. The roots of $x^2 - 6x + 4 = 0$ and $3x^2 - 6x - 1 = 0$ are surds. Do your findings for questions 6 and 7 still hold true?

Find the roots and hence the quadratic equation if

9. $\alpha + \beta = 1$ and $\alpha\beta = -2$
10. $\alpha + \beta = \frac{7}{2}$ and $\alpha\beta = \frac{3}{2}$

Extension

If the discriminant of a quadratic equation is less than zero then it is still possible to find the roots. The Swiss mathematician Leonhard Euler [1707 – 1783] decided to let the $\sqrt{-1} = i$ and called it an *imaginary* number.

Hence

$$i^2 = -1$$

$$\sqrt{-4} = \sqrt{4 \times -1} = \sqrt{4} \times \sqrt{-1} = 2i$$

$$\sqrt{-2} = \sqrt{2}i$$

Euler combined real numbers with imaginary numbers to create *complex* numbers. Euler could not envisage that complex numbers would be used extensively in electronic engineering, airplane and car design. $-3 + 2i$ and $2 - 5i$ are examples of complex numbers.

$$(-3 + 2i) + (2 - 5i) = -1 - 3i$$

$$(-3 + 2i) \times (2 - 5i) = -6 + 15i + 4i - 10i^2 = -6 + 19i - 10 \times (-1) = 4 + 19i$$

Is the connection between the coefficients a, b and c, the sum $\alpha + \beta$ and the product $\alpha\beta$ still true for a quadratic equation with complex roots like $x^2 + 4x + 6 = 0$

Teacher notes

1. $x^2 + 6x + 8 = 0$

   $(x + 2)(x + 4) = 0$

   The roots are -2 and -4

2. $x^2 - 5x + 6 = 0$

   $(x - 2)(x - 3) = 0$

   The roots are 2 and 3

3. $2x^2 + 7x - 15 = 0$

   $(2x - 3)(x + 5) = 0$

   The roots are $\frac{3}{2}$ and -5

4. $3x^2 - 11x + 10 = 0$

   $(3x - 5)(x - 2) = 0$

   The roots are $\frac{5}{3}$ and 2

5. $x^2 - 16 = 0$

   $(x - 4)(x + 4) = 0$

   The roots are 4 and -4

6. $\alpha + \beta = \frac{-b}{a}$ and $\alpha\beta = \frac{c}{a}$
7. $b^2 - 4ac$ is a square number
8. $\alpha + \beta$ is still equal to $\frac{-b}{a}$ and $\alpha\beta$ is till equal to $\frac{c}{a}$ but the discriminant is no longer a square number
9. $\alpha + \beta = 1$ and $\alpha\beta = -2$

   $\alpha = 1 - \beta$

   $\Rightarrow (1 - \beta)\beta = -2$

   $\Rightarrow \beta^2 - \beta - 2 = 0$

   $\Rightarrow (\beta - 2)(\beta + 1) = 0$

   $\Rightarrow \beta = 2$ and $\alpha = -1$ *or* $\beta = -1$ and $\alpha = 2$

   Either way, the roots are 2 and -1

   $\Rightarrow (x - 2)(x + 1) = 0$

   $\Rightarrow x^2 - x - 2 = 0$

10. $\alpha + \beta = \frac{7}{2}$ and $\alpha\beta = \frac{3}{2}$

$$\alpha = \frac{7}{2} - \beta$$

$$\Rightarrow \left(\frac{7}{2} - \beta\right)\beta = \frac{3}{2}$$

$$\Rightarrow (7 - 2\beta)\beta = 3$$

$$\Rightarrow 2\beta^2 - 7\beta + 3 = 0$$

$$\Rightarrow (2\beta - 1)(\beta - 3) = 0$$

$\Rightarrow \beta = \frac{1}{2}$ and $\alpha = 3$ or $\beta = 3$ and $\alpha = \frac{1}{2}$

Either way, the roots are 3 and $\frac{1}{2}$

$$\Rightarrow (x - 3)\left(x - \frac{1}{2}\right) = 0$$

$$\Rightarrow (x - 3)(2x - 1) = 0$$

$$\Rightarrow 2x^2 - 7x + 3 = 0$$

Extension

www.storyofmathematics.com/18th_euler.html

$$x^2 + 4x + 6 = 0$$

$$(x + 2)^2 - 4 + 6 = 0$$

$$(x + 2)^2 + 2 = 0$$

$$(x + 2)^2 = -2$$

$$x + 2 = \pm\sqrt{-2}$$

$$x = -2 \pm \sqrt{-2} = -2 \pm \sqrt{2}i$$

Let $\alpha = -2 + \sqrt{2}i$ and $\beta = -2 - \sqrt{2}i$

$\alpha + \beta = -4 = \frac{-4}{1}$ which is $\frac{-b}{a}$

$\alpha\beta = (= -2 + \sqrt{2}i)(-2 - \sqrt{2}i) = 4 - 2i^2 = 4 + 2 = 6 = \frac{6}{1}$ which is $\frac{c}{a}$