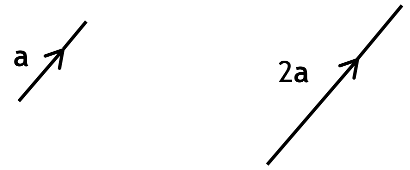
Vectors

A vector represents a journey of a fixed length in a specified direction.
Here is the vector **a**, so the journey **2a** will be a journey twice as long but in the same direction.



These vectors will be parallel as one is a multiple of the other

Parallel vectors

Vectors must be parallel if one is a multiple of the other. So all we need to do if we are asked to prove vectors that are parallel is to use algebra to show that one vector is a multiple of the other, we may need to factorise the algebra to do this.

a. $\mathbf{b}$ and $3\mathbf{b}$ will be parallel

b. $\mathbf{a}$ and $\frac{1}{2}\mathbf{a}$ will be parallel

c. $\mathbf{a} + \mathbf{b}$ and $2\mathbf{a} + 2\mathbf{b}$
will be parallel as $2\mathbf{a} + 2\mathbf{b} = 2(\mathbf{a} + \mathbf{b})$

d. $\mathbf{a} + 3\mathbf{b}$ and $\frac{2}{3}(\mathbf{a} + 3\mathbf{b})$ will be parallel

e. Will $\mathbf{a} + \frac{3}{5}\mathbf{b}$ be parallel to $5\mathbf{a} + 3\mathbf{b}$?

$$\mathbf{a} + \frac{3}{5}\mathbf{b} = \frac{5}{5}\mathbf{a} + \frac{3}{5}\mathbf{b}$$

$$= \frac{1}{5}(5\mathbf{a} + 3\mathbf{b})$$

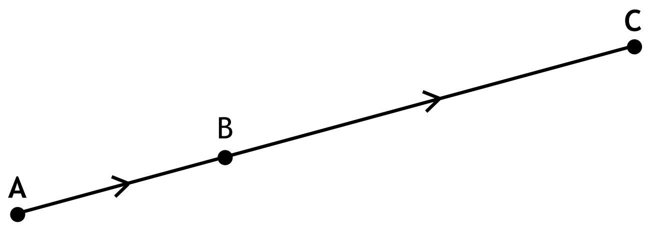
$\mathbf{a}$ can be written as $\frac{5}{5}\mathbf{a}$

They are parallel as one is a multiple of the other

Vectors and a straight line

To show vectors form a single straight line all we need to do is show that the 2 vectors are parallel (one will be a multiple of the other) and that they have a common point.
(They may start at the same point, or one may start where the other ends)

(If 2 vectors start at the same point and are parallel then one must be on top of the other, but one will just be longer, hence they will be on a straight line)



If we know the vector $\overrightarrow{AB} = \mathbf{a} + 2\mathbf{b}$, to prove A,B and C are on a straight line we will need to know either $\overrightarrow{AC}$ or $\overrightarrow{BC}$.

If we find $\overrightarrow{AC} = 3\mathbf{a} + 6\mathbf{b}$

$$\therefore \overrightarrow{AC} = 3(\mathbf{a} + 2\mathbf{b}) = 3\overrightarrow{AB}$$

$\therefore$ the vectors are parallel.

Also as they have a common point at A

they form a straight line

If we find $\overrightarrow{BC} = 2\mathbf{a} + 4\mathbf{b}$

$$\therefore \overrightarrow{BC} = 2(\mathbf{a} + 2\mathbf{b}) = 2\overrightarrow{AB}$$

$\therefore$ the vectors are parallel.

Also as they have a common point at B

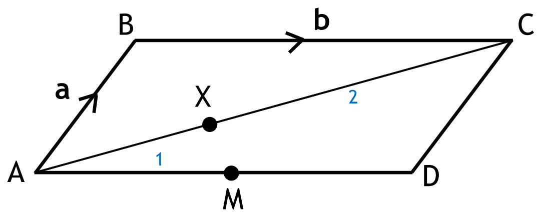
they form a straight line

Examples

1. The diagram shows the parallelogram ABCD. M is the midpoint of AD and X the point on the line AC such that the ratio AX : XC = 1 : 2

$\overrightarrow{AB} = \mathbf{a}$

$\overrightarrow{BC} = \mathbf{b}$



a. $\overrightarrow{BA} = -\mathbf{a}$

b. $\overrightarrow{AD} = \mathbf{b}$

c. $\overrightarrow{AC} = \mathbf{a} + \mathbf{b}$

d. $\overrightarrow{AM} = \frac{1}{2}\mathbf{b}$

e. $\overrightarrow{DB} = -\mathbf{b} + \mathbf{a}$ (or $\mathbf{a} - \mathbf{b}$)

f. $\overrightarrow{BM} = -\mathbf{a} + \frac{1}{2}\mathbf{b}$ (or $\frac{1}{2}\mathbf{b} - \mathbf{a}$)

g. $\overrightarrow{AX} = \frac{1}{3}\overrightarrow{AC} = \frac{1}{3}(\mathbf{a} + \mathbf{b})$

h. $\overrightarrow{BX} = \overrightarrow{BA} + \overrightarrow{AX}$

$= -\mathbf{a} + \frac{1}{3}(\mathbf{a} + \mathbf{b}) = -\mathbf{a} + \frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$

$= -\frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b} = \frac{1}{3}(-2\mathbf{a} + \mathbf{b})$

i. $\overrightarrow{XD} = \overrightarrow{XA} + \overrightarrow{AD}$

$= -\frac{1}{3}(\mathbf{a} + \mathbf{b}) + \mathbf{b}$

$= -\frac{1}{3}\mathbf{a} - \frac{1}{3}\mathbf{b} + \mathbf{b} = -\frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}$

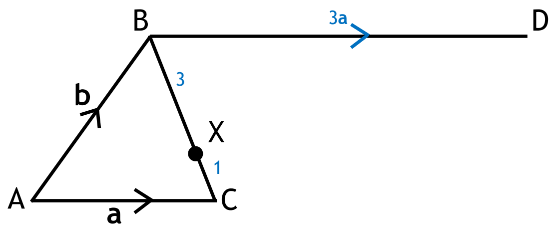
$= \frac{1}{3}(-\mathbf{a} + 2\mathbf{b})$ or $\frac{1}{3}(2\mathbf{b} - \mathbf{a})$

2. The diagram shows the triangle ABC.
X the point on the line CB such that the ratio CX : CB = 1 : 4

$\overrightarrow{AC} = \mathbf{a}$

$\overrightarrow{AB} = \mathbf{b}$

$\overrightarrow{BD} = 3\overrightarrow{AC}$



Show that A, X and D lie on a straight line.

$\overrightarrow{CX} = \frac{1}{4}\overrightarrow{CB} = \frac{1}{4}(-\mathbf{a} + \mathbf{b}) = -\frac{1}{4}\mathbf{a} + \frac{1}{4}\mathbf{b}$

$\overrightarrow{AD} = \mathbf{b} + 3\mathbf{a}$ or $3\mathbf{a} + \mathbf{b}$

$\overrightarrow{AX} = \overrightarrow{AC} + \overrightarrow{CX}$

$\overrightarrow{AX} = \mathbf{a} - \frac{1}{4}\mathbf{a} + \frac{1}{4}\mathbf{b} = \frac{3}{4}\mathbf{a} + \frac{1}{4}\mathbf{b}$

$= \frac{1}{4}(3\mathbf{a} + \mathbf{b}) = \frac{1}{4}\overrightarrow{AD}$

$\overrightarrow{AX}$ and $\overrightarrow{AD}$ are parallel as one is a multiple of the other. They also have a common point at A so A ,X and D are all on the same straight line.

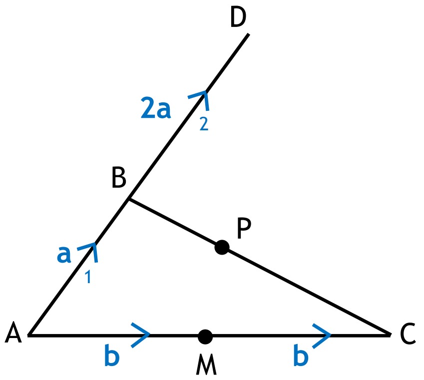
3. The diagram shows the triangle ABC.
M is the midpoint of AC.
ABD is a straight line, and the ratio AB : BD is 1 : 2.
P is a point on the line BC.

A harder question

$\overrightarrow{AB} = \mathbf{a}$

$\overrightarrow{AC} = 2\mathbf{b}$

$\overrightarrow{BP} = k\overrightarrow{BC}$



Given that DPM is a straight line, find the value of k.

Using the line BPC

$\overrightarrow{BC} = -\mathbf{a} + 2\mathbf{b}$

$\overrightarrow{BP} = k\overrightarrow{BC} = k(-\mathbf{a} + 2\mathbf{b}) = -k\mathbf{a} + 2k\mathbf{b}$

Using the line DPM

$\overrightarrow{DP} = \overrightarrow{DB} + \overrightarrow{BP}$

$= -2\mathbf{a} - k\mathbf{a} + 2k\mathbf{b}$

$\overrightarrow{DM} = -3\mathbf{a} + \mathbf{b}$

As DPM is a straight line $\overrightarrow{DM}$ is a multiple of $\overrightarrow{DP}$. Let us say $\overrightarrow{DM} = n\overrightarrow{DP}$

$\therefore -3\mathbf{a} + \mathbf{b} = n(-2\mathbf{a} - k\mathbf{a} + 2k\mathbf{b})$

Now we need to equate coefficients.
The coefficient of **a** = -3
The coefficient of **b** = 1

$-3\mathbf{a} + \mathbf{b} = -2n\mathbf{a} - kn\mathbf{a} + 2kn\mathbf{b}$

$= \mathbf{a}(-2n - kn) + \mathbf{b}(2kn)$

$\therefore 2kn = 1$ and $-2n - kn = -3$

$kn = \frac{1}{2}$ $\quad \therefore \quad -2n - \frac{1}{2} = -3$

$-2n = -2.5$

$n = \frac{-2.5}{-2}$

$n = 1.25$ (or $n = \frac{5}{4}$)

$kn = \frac{1}{2}$

$\frac{5}{4}k = \frac{1}{2}$

$k = \frac{4}{10}$ $\quad \therefore$ $\mathbf{k = \frac{2}{5}}$ or $\mathbf{k = 0.4}$