Quadratics: Solve, sketch graphs and determine the turning points

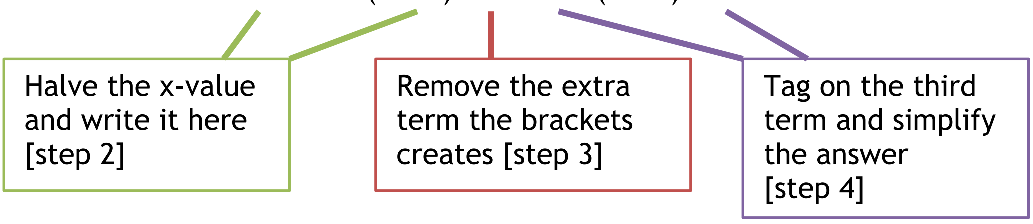
The following steps can be used to 'Complete the square' for $ax^2 + bx + c$

1. Make sure your quadratic starts with … x^2. If it has ax^2 in it then divide throughout by a and put a as a multiplier in front of the bracket (see e.g.2)
2. Halve the value of the **x-term** and write this answer in the bracket such as $(x - *)^2$
3. Square the * value and **subtract** it from the squared bracket. This removes the extra term the bracket would create.
4. Tag on the remaining third term and simplify the resulting sum.

e.g.1. When $a = 1$ (start at step 2)

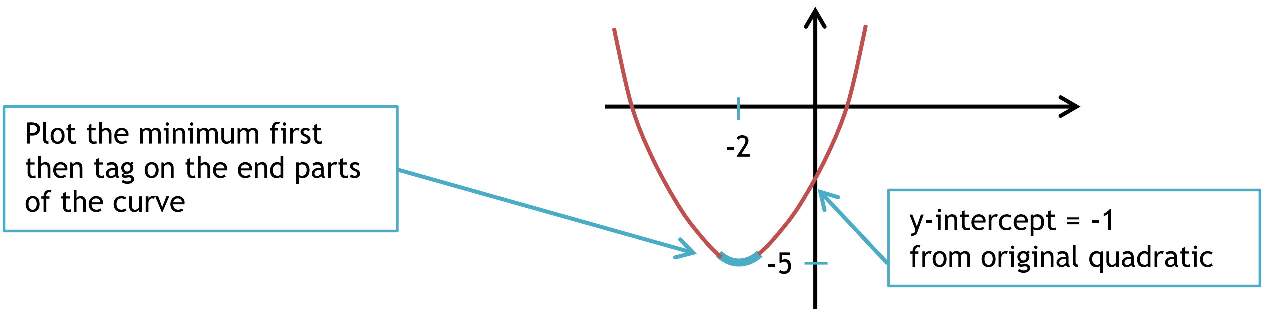
Find the coordinates of the minimum point and sketch the curve for $y = x^2 + 4x - 1$.

$$x^2 + 4x - 1 = (x + 2)^2 - 2^2 - 1 = (x + 2)^2 - 5$$



Now we know that $y = (x + 2)^2 - 5$, the minimum value is -5 because the bracket is squared so must be positive. This will occur when the $x + 2 = 0$. So, $x = -2$ and the minimum point is (-2, -5).

Plotting the graph



If we set f(x) = 0, we can solve the quadratic equation to find where the curve would cross the x-axis.

$$\Rightarrow x^2 + 4x - 1 = 0 \Rightarrow (x + 2)^2 - 5 = 0$$
$$\Rightarrow (x + 2)^2 = 5$$
$$\Rightarrow x + 2 = \pm\sqrt{5}$$
$$\therefore x = -2 \pm \sqrt{5}$$

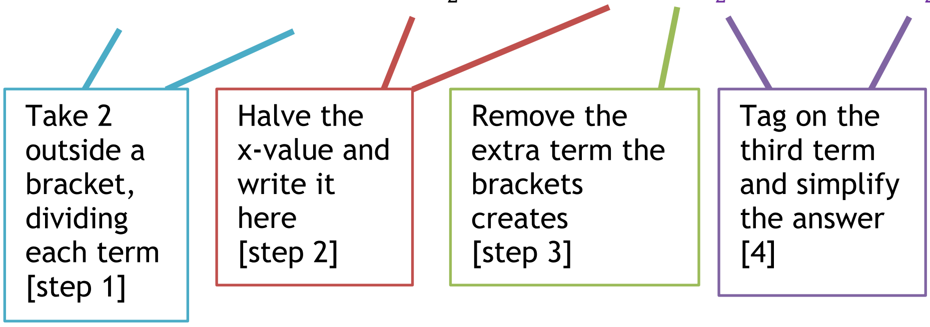
Exercise 1: for each of the following draw a sketch of each function by completing the square and showing where the curve crosses the co-ordinate axes.

1. $y = x^2 + 4x - 5$	**2.** $y = x^2 - 4x - 5$	**3.** $y = x^2 + 6x - 11$	**4.** $y = x^2 - 6x - 5$
5. $y = x^2 - 8x + 3$	**6.** $y = x^2 + 10x + 8$	**7.** $y = x^2 + 8x - 3$	**8.** $y = x^2 - 10x - 7$

e.g.2. When $a > 1$ (start at step 1)

Find the coordinates of the minimum point and sketch the curve for $y = 2x^2 + 4x - 3$.

$$2x^2 - 4x - 3 = 2[x^2 - 2x - \frac{3}{2}] = 2[(x - 1)^2 - 1^2 - \frac{3}{2}] = 2[(x - 1)^2 - \frac{5}{2}] = \underline{\underline{2(x - 1)^2 - 5}}$$



Plotting the graph

From completing the square, we can see the coordinates of the minimum point are <u>(1, -5)</u>

Solve $2x^2 - 4x - 3 = 0$ to find where the graph crosses the x-axis.

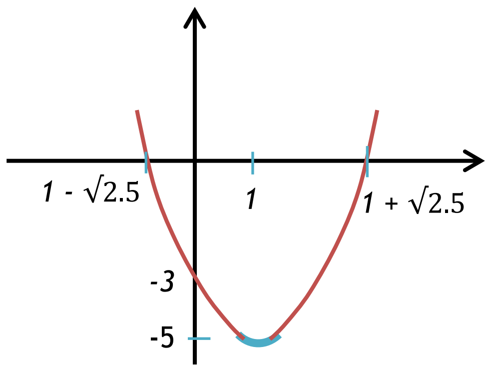
$\Rightarrow 2(x - 1)^2 - 5 = 0$

$\Rightarrow \quad 2(x - 1)^2 = 5$

$\Rightarrow \quad (x - 1)^2 = 2.5$

$\Rightarrow \quad x - 1 = \pm \sqrt{2.5}$

$\therefore x = 1 \pm \sqrt{2.5}$



e.g.3. Sketch the curve of $y=3x^2 + 12x + 14$

Step 1: $y = 3(x^2 + 4x + \frac{14}{3})$

Step 2+3: $y = 3[(x + 2)^2 - 2^2 + \frac{14}{3}]$

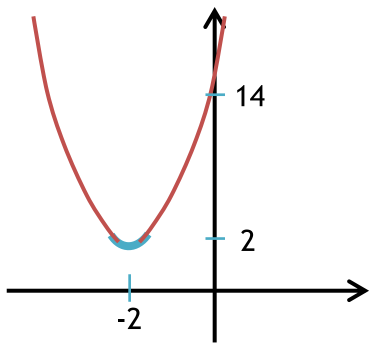
Step 4: $y = 3[(x + 2)^2 - 4 + \frac{14}{3}]$

$y = 3[(x + 2)^2 + \frac{2}{3}]$

$y = 3(x + 2)^2 + 2$

Minimum point = <u>(-2, 2)</u>

The graph does not cross the x-axis. What happens if you solve f(x) = 0?



Exercise 2: for each of the following draw a sketch of each function by completing the square and showing where the curve crosses the co-ordinate axes.

1. $y = 2x^2 - 4x - 5$

2. $y = 2x^2 - 12x + 17$

3. $y = 2x^2 + 4x - 2$

4. $y = 2x^2 + 8x + 5$

5. $y = 2x^2 - 8x + 15$

6*. $y = 7 + 4x - x^2$

* multiply by '-1' first by taking it out as a common factor (this changes **all** the signs)

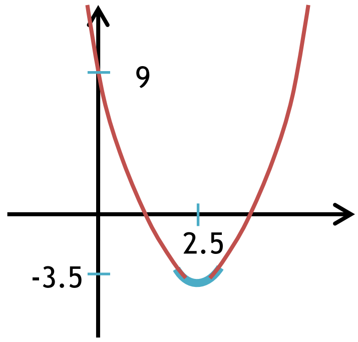
e.g.4. Completing the Square and Transformations of the Quadratic function

We can use the final completed square expression to transform the basic quadratic curve **$y = x^2$**

$y = 2x^2 - 10x + 9 = 2\left(x - \frac{5}{2}\right)^2 - \frac{7}{2}$.

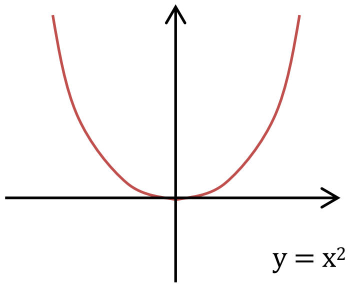
So the minimum point = $\left(\frac{5}{2}, -\frac{7}{2}\right)$.

The sketch would look like this . . . .

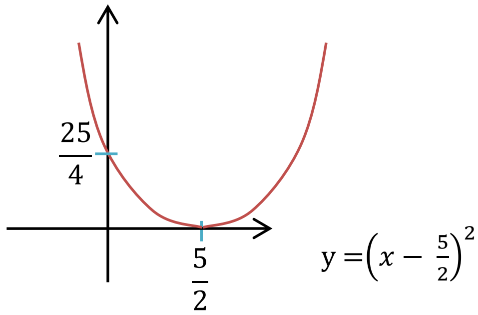


This could be achieved by adopting the following three transformations.

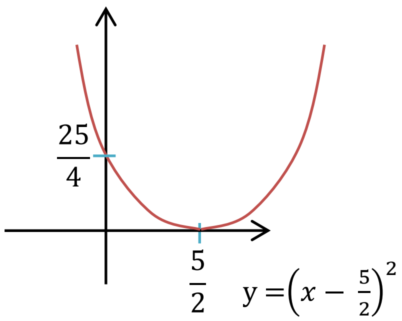
1. Translate (move) the $y=x^2$ curve '$+\frac{5}{2}$' to the right. This give the curve $y = \left(x - \frac{5}{2}\right)^2$



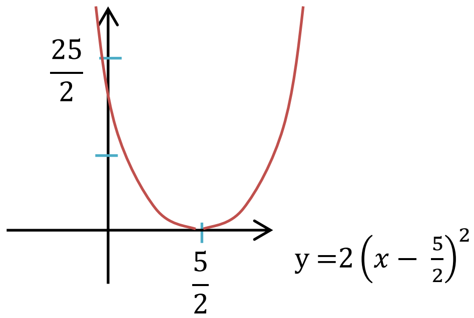
$\frac{5}{2}$ units to the right



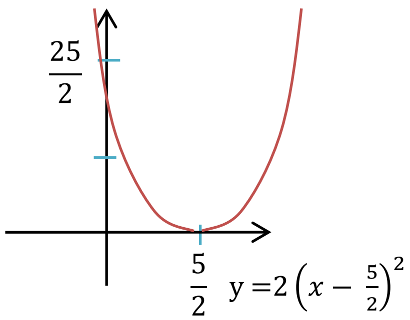
2. Multiply up the y-axis by a scale factor of **2**. This doubles all the y-coordinates and give the curve $y = \mathbf{2}\left(x - \frac{5}{2}\right)^2$



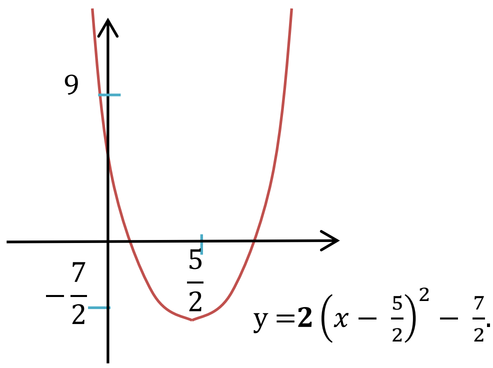
Multiply up by 2



3. Translate (move) the curve '$-\frac{7}{2}$' down the y-axis.



Move down $\frac{7}{2}$ units

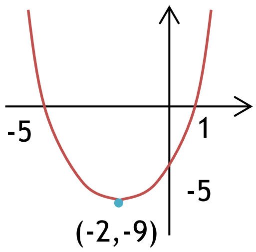


Teaching notes: answers

Exercise 1:

1. $(x + 2)^2 - 9$; Min = (-2, -9); crosses x-axis at (-5, 0) & (1, 0); crosses y-axis at (0, -5).
2. $(x - 2)^2 - 9$; Min = (2, -9); crosses x-axis at (-1, 0) & (5, 0); crosses y-axis at (0, -5).
3. $(x + 3)^2 - 20$; Min = (-3, -20); crosses x-axis at (-7.47, 0) & (1.47, 0); crosses y-axis at (0, -11).
4. $(x - 3)^2 - 14$; Min = (3, -14); crosses x-axis at (-0.74, 0) & (6.74, 0); crosses y-axis at (0, -5).
5. $(x - 4)^2 - 13$; Min = (4, -13); crosses x-axis at (0.39, 0) & (7.61, 0); crosses y-axis at (0, 3).
6. $(x + 5)^2 - 17$; Min = (-5, -17); crosses x-axis at (-9.12, 0) & (-0.88, 0); crosses y-axis at (0, 8).
7. $(x + 4)^2 - 19$; Min = (-4, -19); crosses x-axis at (-8.36, 0) & (0.36, 0); crosses y-axis at (0, -3).
8. $(x - 5)^2 - 32$; Min = (5, -32); crosses x-axis at (-0.66, 0) & (10.66, 0); crosses y-axis at (0, -7).

1.



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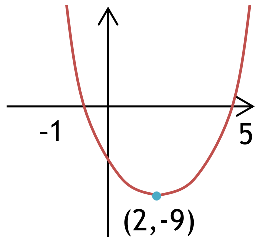

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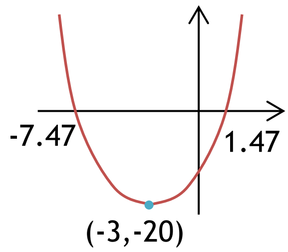

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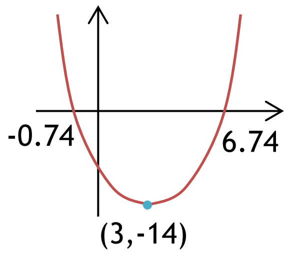

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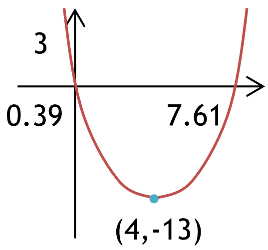

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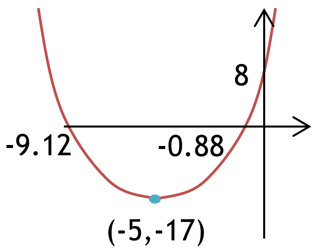

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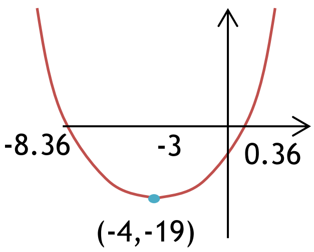

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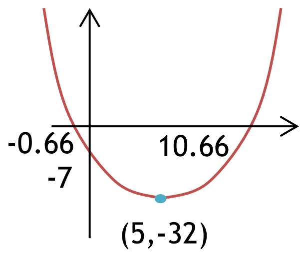
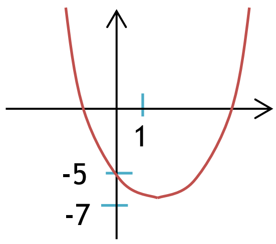
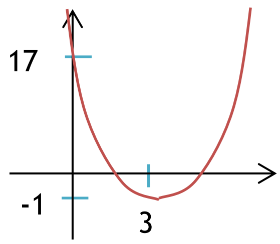


Exercise 2:

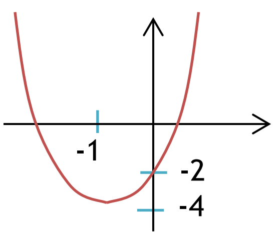
1. $2(x - 1)^2 - 7$
Min = (1, -7)
y-int = (0,-5)
x-ints =(2.87,0)
(-0.87,0)



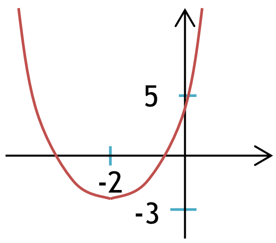
2. $2(x - 3)^2 - 1$
Min = (3, -1)
y-int = (0,17)
x-ints=(2.29,0)
(3.71,0)



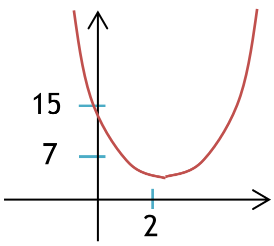
3. $2(x + 1)^2 - 4$
Min = (-1, -4)
y-int = (0,-2)
x-ints =(-2.41,0)
(0.41,0)



4. $2(x + 2)^2 - 3$
Min = (-2, -3)
y-int = (0,5)
x-int=(-3.22,0)
(-0.78,0)



5. $2(x - 2)^2 + 7$
Min = (2, 7)
y-int = (0,15)



6. $11 - (x - 2)^2$
MAX = (2, 11)
y-int = (0,7)
x-int=(-1.32,0)
(5.32,0)